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SELF-SIMILAR SOLUTIONS OF THE PROBLEM OF DISPLACEMENT OF ONE GAS BY ANOTHER IN AN AXISYMMETRIC CASE WITH A QUADRATIC DRAG LAW

Yu. N. Gordeev,[1] **A. E. Sandakov,**[1] **and Yu. L. Chizhov**[2]

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A problem of piston-induced displacement of one gas by another in cracks (porous media) in an axisymmetric case with a quadratic drag law is studied. Self-similar solutions for determining the dynamic characteristics (velocity and pressure) of the displacing and displaced gases are constructed in quadratures. The velocity and pressure are studied as functions of a self-similar variable for several initial conditions and parameters.

Key words: *self-similarity, displacement, gas, drag, crack, porous medium.*

An exact solution of the problem of a gas flow in a porous medium in a two-dimensional case was obtained in [1, 2]. The problem of displacement of one liquid by another in such a case was considered in [3, 4].

At present, the study of the gas flow is an urgent problem in the theory of hydrofracturing (see, e.g., [5, 6]).

In the present paper, we consider the problem of isothermal displacement of one gas by another in well-permeable cracks in an axisymmetric flow with a quadratic drag law. Self-similar solutions for determining the velocity and pressure of the displacing and displaced gases are constructed in quadratures. Self-similar velocity and pressure are plotted as functions of a self-similar variable for several initial conditions and parameters.

1. Formulation of the Problem of Displacement of One Gas by Another with a Quadratic Drag Law. For high flow velocities, the equations of isothermal motion of the gas in a channel (or in a porous medium) have the form [7–12]

$$\frac{\partial \rho}{\partial t} + \frac{1}{x^m}\frac{\partial}{\partial x}(x^m u\rho) = 0,$$

$$\frac{\partial p}{\partial x} = -\frac{b}{w^2}u^2\rho \qquad (p = c^2\rho),$$

where ρ is the density, u is the velocity, p is the gas pressure, x is the coordinate, t is the time, m is the index of symmetry of the problem, c is the isothermal velocity of sound in the gas, w is the crack angle, and b is an experimental constant determined by the crack roughness and by the Reynolds number.

In the case of an axisymmetric flow ($m = 1$), under the assumption that only one species is present at each particular point, the equations of gas displacement have the form [12]

$$\frac{\partial}{\partial t}\rho_i + \frac{1}{x}\frac{\partial}{\partial x}(xu_i\rho_i) = 0, \qquad \frac{\partial}{\partial x}p_i = -\frac{b}{w^2}u_i^2\rho_i, \qquad p_i = c_i^2\rho_i. \tag{1.1}$$

Here subscript $i = 1$ refers to the displacing gas $[0 \leqslant x \leqslant L(t)]$ and $i = 2$ refers to the displaced gas $[L(t) \leqslant x < +\infty]$, where $L(t)$ is the interface between the gases].

The interface position $L(t)$ under given boundary and initial conditions is determined by solving system (1.1).

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[1] Moscow Engineering Physics Institute (State University), Moscow 115409. [2] Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow 119526; sandakovanton@mail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 5, pp. 87–92, September–October, 2008. Original article submitted May 22, 2007; revision submitted September 13, 2007.

Assuming that the displacement is induced by a piston, we define the initial and boundary conditions of the problem as

$$p(0,t) = p^*, \qquad p(x,0) = p_0. \tag{1.2}$$

For the interface between the gases $x = L(t)$, the Hugoniot conditions (continuity of the mass and momentum fluxes) yield

$$p_1(L(t),t) = p_2(L(t),t), \qquad p_1(u_1 - D)\Big|_{x=L(t)} = p_2(u_2 - D)\Big|_{x=L(t)} \tag{1.3}$$

($D = dL/dt$ is the interface velocity).

The mass flux of the gas through the interface of two immiscible gases equals zero; hence, conditions (1.3) acquire the form $u_1(L(t),t) = u_2(L(t),t) = D$.

Eliminating the density ρ_i from Eqs. (1.1), we obtain

$$\frac{\partial p}{\partial t} + \frac{1}{x}\,\frac{\partial}{\partial x}\,(xup) = 0, \qquad \frac{\partial p}{\partial x} = -\frac{u^2 p}{l}, \tag{1.4}$$

where $l = l_1 = w^2/(bc_1^2)$, $p = p_1$, and $u = u_1$ for $0 \leqslant x \leqslant L(t)$ and $l = l_2 = w^2/(bc_2^2)$, $p = p_2$, and $u = u_2$ for $L(t) \leqslant x < +\infty$. The initial and boundary conditions (1.2) and (1.3) are written in the form

$$\begin{gathered} p(x,0) = p_0, \qquad p(0,t) = p^*, \\ p(L(t)-0,t) = p(L(t)+0,t), \qquad u(L(t)-0,t) = u(L(t)+0,t) = D. \end{gathered} \tag{1.5}$$

Problem (1.4), (1.5) is a self-similar problem with the variable

$$\theta = x(4/(9t^2 l_1))^{1/3} = (4/(9l_1))^{1/3} x t^{-2/3}.$$

The pressure p and velocity u of the gas flow in the channel are expressed via their dimensionless analogs $f(\theta)$ and $\varphi(\theta)$:

$$p = p^* f(\theta), \qquad u = (2l_1/(3t))^{1/3}\varphi(\theta).$$

The self-similar variable θ_0 corresponding to the interface coordinate is $\theta_0 = L(t)[4/(9t^2 l_1)]^{1/3}$.

In the self-similar variables, problem (1.4) becomes

$$\varphi' - \varkappa\varphi^3 + \theta\varkappa\varphi^2 + \varphi/\theta = 0, \qquad f' + \varkappa\varphi^2 f = 0, \tag{1.6}$$

where

$$\varkappa = \begin{cases} 1, & 0 \leqslant \theta \leqslant \theta_0, \\ \varkappa_0 = c_1^2/c_2^2, & \theta_0 < \theta < +\infty; \end{cases}$$

and the prime denoted the derivative with respect to the variable θ.

Taking into account that $\lim\limits_{x\to\infty}(xu\rho) = A$, where $A = \text{const}$, in this formulation, we can write the initial and boundary conditions (1.2) in the self-similar variables as

$$\lim_{\theta\to 0}(f\theta\varphi) = 1, \qquad \lim_{\theta\to\infty} f = N = p_0/p^*, \tag{1.7}$$

and the conditions on the interface between the gases as

$$f(\theta_0 - 0) = f(\theta_0 + 0), \qquad \varphi(\theta_0 - 0) = \varphi(\theta_0 + 0) = \theta_0. \tag{1.8}$$

2. Self-Similar Solutions of the Displacement Problem. Renormalizing the first equation in (1.6) as

$$\varphi = (2/3)^{-1/3}\varkappa^{-1/3}\eta, \qquad \theta = (2/3)^{2/3}\varkappa^{-1/3}\xi, \tag{2.1}$$

we can reduce it to an equation of the form

$$\frac{d\eta}{d\xi} - \eta^3 + \frac{2}{3}\,\xi\eta^2 + \frac{\eta}{\xi} = 0. \tag{2.2}$$

Using the substitutions $1/\eta = (2/3)\chi\xi^2$ and $\chi = 1 - 3^{2/3}\psi/\xi$, we obtain an equation with respect to the function $\xi(\psi)$:

$$\frac{d\xi}{d\psi} = 4 \cdot 3^{-4/3}\xi^2(\xi - 3^{2/3}\psi). \tag{2.3}$$

It follows from Eq. (2.3) that $\xi(\psi)$ is a strictly monotonic function; hence, there exists an inverse function $\psi(\xi)$.

Passing to a new variable z by the formula $\xi = 3^{2/3}\cdot 2^{-1}(\psi^2 - z)^{-1}$ and using the expression $\eta = (3/2)\xi^{-2}(1 - 3^{2/3}\psi/\xi)^{-1}$, we obtain the equation $d\psi/dz = \psi^2 - z$. With the substitution $\psi = -y'/y$, the latter equation is reduced to Airy's equation

$$y'' = zy. \tag{2.4}$$

Airy's equation (2.4) has a general solution of the form $y = D_1A_i(z) + D_2B_i(z)$, where $A_i(z)$ and $B_i(z)$ are Airy's functions [13, 14]; D_1 and D_2 are constants.

In our case, Airy's equation (2.4) has the following solutions:

$$y(z) = \begin{cases} D_1A_i(z) + D_2B_i(z), & 0 \leqslant \theta < \theta_0, \\ D_3A_i(z) + D_4B_i(z), & \theta_0 < \theta. \end{cases}$$

The first solution describes the displacing gas with a parameter $\varkappa = 1$, and the second solution describes the displaced gas with a parameter $\varkappa = \varkappa_0 = c_1^2/c_2^2$. The constants D_i $(i = 1, \ldots, 4)$ are determined from the boundary conditions of the problem.

From the second equation in (1.6) and Eq. (2.1), we obtain

$$\frac{df}{d\xi} + \eta^2 f = 0. \tag{2.5}$$

Integrating Eq. (2.5) and using the chain of equalities obtained in [12], we can write $f = (2/3)y^2 + (4/3)y'^3y^{-1} - (4/3)y'yz$. Thus, the gas pressure f is expressed via the function $y(z)$.

The variables ξ and z are related by the function $\psi(z)$:

$$y'^2y^{-2} = 3^{2/3}\cdot 2^{-1}\xi^{-1} + z. \tag{2.6}$$

Let us consider the asymptotic solutions of Eq. (2.2). As $\xi \to \infty$, the gas velocity is $\eta(\xi) \to 0$. In this case, the asymptotic solution of Eq. (2.2) has the form $\eta \approx (3/2)\xi^{-2} + E\xi^{-3} + \ldots$, where E is a constant. It follows from the definition of the function ψ that ψ tends to a constant value as $\xi \to \infty$, and $\psi^2(z_1) = z_1$.

For $\xi \to 0$, the asymptotic solutions of Eq. (2.2) can be presented in the form $\eta \approx 1/\sqrt{2\xi}$ and $\psi \approx -3^{-1/3}(2\xi)^{-1/2}$, i.e., $\psi \to -\infty$ as $\xi \to 0$.

As $\psi = -y'y^{-1}$, the point $\xi = 0$ corresponds to the point $z = z_0$ [$z_0 < z_1$ is the point with $y(z_0) = 0$ closest to z_1].

As ξ changes from zero to ∞, the variable z changes from z_0 to z_1 with the only discontinuity at the interface between the gases.

For $\xi \to 0$ or $z = z_0$, we have

$$\lim_{\theta\to 0}(f\theta\varphi) = 1, \qquad y(z_0) = 0.$$

Using the identity $A_i'(z_0)B_i(z_0) - A_i(z_0)B_i'(z_0) = 1/\pi$ [14], we obtain

$$D_1 = 3^{1/2}\cdot 2^{-2/3}\pi B_i(z_0), \qquad D_2 = -3^{1/2}\cdot 2^{-2/3}\pi A_i(z_0).$$

For $\xi \to \infty$ or $z = z_1$, we have

$$\lim_{\xi\to\infty} f = N, \qquad \psi^2(z_1) = z_1,$$

$$D_3 = -\sqrt{1.5N}\,\pi(B_i'(z_1) - \sqrt{|z_1|}\,B_i(z_1)), \qquad D_4 = \sqrt{1.5N}\,\pi(A_i'(z_1) - \sqrt{|z_1|}\,A_i(z_1)).$$

Using Eqs. (2.1) and (2.6) and the definition of the function η, we can write the dependence $z(\theta)$:

$$z = \Big(\frac{\varkappa}{4}\Big)^{2/3}\Big(\theta - \frac{1}{\varkappa\varphi\theta}\Big)^2 - \frac{1}{\theta}(2\varkappa)^{-1/3}, \qquad \varkappa = \begin{cases} 1, & 0 \leqslant \theta \leqslant \theta_0, \\ \varkappa_0, & \theta_0 < \theta < +\infty. \end{cases}$$

At the point θ_0, the gas velocity $\varphi(\theta)$ is continuous, and the parameter $\varkappa$ changes in a jumplike manner from $\varkappa = 1$ to $\varkappa = \varkappa_0$; hence, the function $z(\theta)$ also changes at the point θ_0 in a jumplike manner from $z(\theta_0 - 0) = z^-$ to $z(\theta_0 + 0) = z^+$, where

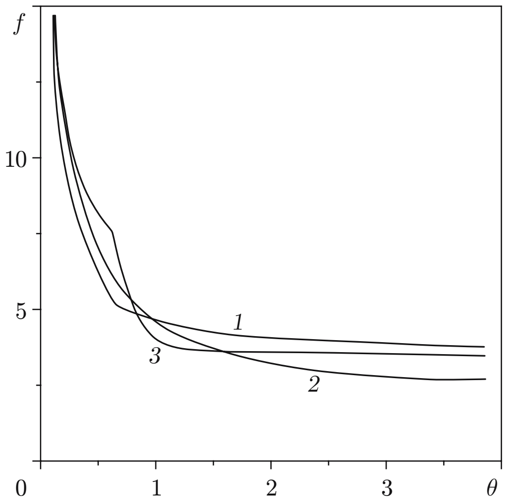

Fig. 1

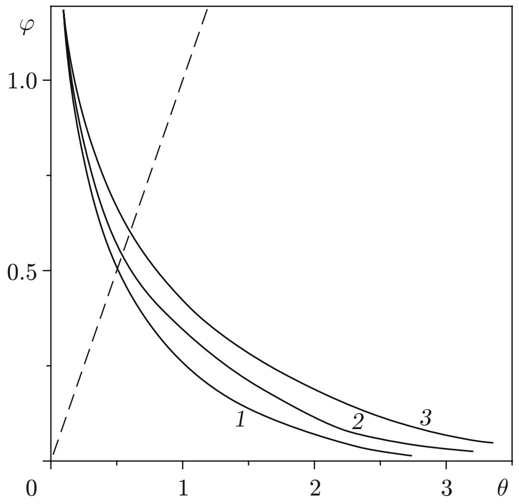

Fig. 2

Fig. 1. Gas pressure f versus the self-similar variable θ: 1) $\varkappa_0 = 0.5$ and $N = 3.1$; 2) $\varkappa_0 = 1$ and $N = 2.85$; 3) $\varkappa_0 = 4$ and $N = 3.5$.

Fig. 2. Gas velocity φ versus the self-similar variable θ: the dashed curve shows the data for $\varphi = \theta$; the remaining notation is the same as in Fig. 1.

$$z^- = 2^{-1/3}(\theta_0^2/(2(1-\theta_0^{-3})^2) - \theta_0^{-1}), \qquad z^+ = (\varkappa_0/4)^{2/3}(\theta_0 - (\varkappa_0\theta_0^2)^{-1})^2 - \theta_0^{-1}(2\varkappa_0)^{-1/3}.$$

In the intervals $\theta \in [0, \theta_0)$ and $\theta \in (\theta_0, +\infty)$, the function $z(\theta)$ behaves monotonically.

Taking into account conditions (1.8) and (2.1) and using the definition of the function η, we obtain

$$y(z^-) = \varkappa_0^{-1/2} y(z^+), \qquad \frac{y'(z^+)}{y'(z^-)} = \varkappa_0^{-1/6}\Big(\frac{1-\theta_0^3\varkappa_0}{1-\theta_0^3}\Big). \tag{2.7}$$

Thus, for all fixed N and $\varkappa_0$ from (1.8) and the condition $\theta(z^-) = \theta(z^+)$, the values of z_0 and z_1 can be determined.

Let us consider the algorithm of constructing a solution of problem (1.6)–(1.8). Note that N and $\varkappa_0$ are fixed parameters of the problem, and z_0 is an auxiliary parameter to be selected. First, a certain value of z_0 is chosen, after which the constants D_1 and D_2 and the function $y(z)$ for the displacing gas are found. Thus, the functions $\varphi(z)$ and $\theta(z)$ for the displacing gas are determined on the interval $z \in [z_0, z^-]$. Based on z_0, the value of z^- is found, and then z^+ is determined with the use of z^-, given $\varkappa_0$, and conditions (1.8). The constants D_3 and D_4 are calculated on the basis of conditions (2.7) on the interface. Thus, the function $y(z)$ and the functions $f(z)$, $\varphi(z)$, and $\theta(z)$ for the displaced gas on the interval $z \in [z^+, z_1]$ are determined. Then the value of z_1 is found, and a certain value of $\tilde{N}$ corresponding to the chosen z_0 is determined. The value of the parameter z_0 corresponding to the given value of the parameter N can be determined by matching with appropriate accuracy.

Dependences of the dynamic characteristics of the gases $f(\theta)$ and $\varphi(\theta)$ as functions of the self-similar variable θ for different initial conditions are constructed.

Figures 1 and 2 show the gas pressure f and gas velocity φ as functions of θ.

It follows from Fig. 1 that a jump in the derivative of pressure $f(\theta)$ corresponds to the interface between the gases $\theta = \theta_0$. If the density of the displaced gas is greater than the density of the displacing gas ($\varkappa_0 > 1$), then the jump of the derivative is negative ($df/d\theta|_{\theta=\theta_0+0} - df/d\theta|_{\theta=\theta_0-0} < 0$); in the case of the smaller density of the displaced gas ($\varkappa_0 < 1$), the jump of the derivative is positive.

The results of the present work can be used to study the gas flow in an axisymmetric crack at high velocities of motion [11] and to test various numerical algorithms.

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